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HEAT KERNEL LOWER GAUSSIAN ESTIMATES IN THE DOUBLING SETTING WITHOUT POINCARÉ INEQUALITY

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Abstract

In the setting of a manifold with doubling property satisfying a Gaussian upper estimate of the heat kernel, one gives a characterization of the lower Gaussian estimate in terms of certain Hölder inequalities.

Contents

Introduction

Let M be a connected non-compact Riemannian manifold without boundary, and let Δ denote the Laplace-Beltrami operator on M. Consider the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0,$$

where $u = u(x,t)$, $x \in M$, $t > 0$.

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The heat kernel $p_t(x,y)$ is by definition the smallest positive fundamental solution to the heat equation on $]0,+\infty[\times M$. In particular, the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u \\ u(0,.) = f(.), \end{cases}$$

where f is a bounded continuous function, is solved by

$$u(t,x) = \int_M p_t(x,y) f(y)\,dy.$$

In the Euclidean space $\mathbb{R}^n$, the heat kernel is given by the following well known formula:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Denote by $B(x,r)$ the ball of center $x \in M$ and radius $r > 0$ with respect to the Riemannian distance d, and by $V(x,r)$ its Riemannian volume. If M is geodesically complete and has non-negative Ricci curvature, Li and Yau (see [**14**]) found that the heat kernel satisfies the two-sided Gaussian estimate:

$$\frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \qquad \forall\, t>0,\ x,y \in M.$$

One says that M satisfies the doubling property if

$$(D) \qquad V(x,2r) \le CV(x,r), \quad \forall\, x \in M,\ r > 0.$$

A consequence of (D) is that there exists $\nu > 0$ such that

$$(D_\nu) \qquad \frac{V(x,s)}{V(x,r)} \le C\left(\frac{s}{r}\right)^\nu, \quad \forall\, x \in M,\ s \ge r > 0.$$

Note that, if $\operatorname{Ricc} M \ge 0$, then M Satisfies the doubling property. We say that M admits the relative Faber-Krahn inequality if for any ball $B(x,r) \subset M$ and any precompact open set $\Omega \subset B(x,r)$

$$(FK) \qquad \lambda_1(\Omega) \ge \frac{b}{r^2}\left(\frac{V(x,r)}{\mu(\Omega)}\right)^\nu,$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue for $-\Delta$, b and ν some positive constants.

On a geodesically complete manifold (see [**9**]), Grigor'yan proved that (FK) is equivalent to the Gaussian upper estimate

$$(UE) \qquad p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \quad t>0,\ x,y\in M,$$

in conjunction with the doubling property (D_ν).

Saloff-Coste (see [**18**], [**19**]) found that the two-sided Gaussian estimate of the heat kernel

$$\frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \qquad t>0,\ x,y\in M,$$

is equivalent to (D) and the 2-Poincaré inequalities

$$(P_2) \qquad \int_{B(x,r)} |f(y)-f_r(x)|^2\, d\mu(y) \le Cr^2 \int_{B(x,Cr)} |\nabla f(y)|^2\, d\mu(y),$$

for all $f\in \mathcal{C}_0^\infty(M)$, $x\in M$, $r>0$, where $f_r(x) = \frac{1}{V(x,r)}\int_{B(x,r)} f(y)\, d\mu(y)$, see also [**8**].

Assuming that the Gaussian upper estimate holds, Coulhon and Ouhabaz gave some simple inequalities (other than Poincaré) that are necessary/sufficient to complete the two-sided Gaussian estimate.

In the case where the volume growth is polynomial, that is

$$V(x,r) \simeq r^D, \quad D>0,$$

Coulhon (see [**5**]) proved that, if (UE) holds, then

$$(LE) \qquad \frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y), \quad t>0,\ x,y\in M,$$

is equivalent to the Sobolev type inequality

$$(S^D) \qquad \frac{|f(x)-f(y)|}{d(x,y)^{\alpha-(D/p)}} \le C\|\Delta^{\alpha/2} f\|_p, \quad \forall\, f\in \mathcal{C}_0^\infty(M),\ x,y\in M,$$

for $p>1$ and some $\alpha > \frac{D}{p}$.

In this polynomial setting and assuming (UE), Ouhabaz (see [**16**]) had given previously a characterization of (LE) in terms of a weaker Gagliardo-Nirenberg type inequality:

$$(G^D) \qquad \frac{|f(x)-f(y)|}{[d(x,y)]^{\alpha-(D/p)}} \le C\|f\|_p^{1-\theta}\|\Delta^{\alpha/2\theta} f\|_p^{\theta}, \quad x,y\in M,$$

for $p>1$ and some $\alpha > \frac{D}{p}$, $\theta \in]0,1[$.

In the more general case when the doubling condition (D_ν) is satisfied, Coulhon proved that, if (UE) holds, then for any $p>1$ and $\alpha>\frac{\nu}{p}$:

$$|f(x)-f(y)| \leq C\frac{d^\alpha(x,y)}{V^{1/p}(x,d(x,y))}\|\Delta^{\alpha/2}f\|_p,$$

implies (LE).

The aim of the present paper is to give similar necessary and sufficient conditions for the lower Gaussian estimate (LE) in the general setting when the volume is only doubling. More precisely, the necessary/sufficient conditions will be given by the following main results:

Theorem. *Assume that (D_ν) and (UE) holds. Then*

$$(LE) \qquad \frac{c}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{ct}\right) \leq p_t(x,y), \quad t>0,\ x,y\in M,$$

implies that for all p large enough there are $\alpha,\alpha'>\frac{\nu}{p}$ such that

$$(S)\ \begin{cases} |f(x)-f(y)| \leq \dfrac{C}{V^{1/p}(x,d)}\max\left\{d^\alpha\|\Delta^{\alpha/2}f\|_p,\ d^{\alpha'}\|\Delta^{\alpha'/2}f\|_p\right\} \\ \forall\ f\in\mathcal{D},\ x,y\in M. \end{cases}$$

Conversely, (S) for any $p>1$ and $\alpha,\alpha'>\frac{\nu}{p}$, implies (LE), where d denote to $d(x,y)$.

Theorem. *Assume that (D_ν) and (UE) holds. Then*

$$(LE) \qquad \frac{c}{V(x,\sqrt{t})}\exp\left(-\frac{d^2(x,y)}{ct}\right) \leq p_t(x,y), \quad t>0,\ x,y\in M,$$

implies for all p large enough, there are $\alpha,\alpha'>\frac{\nu}{p}$ and $\theta,\theta'\in]0,1[$ such that $\alpha-\alpha'=\frac{\nu}{p}$, $\frac{\alpha}{\theta}=\frac{\alpha'}{\theta'}$ and

$$(G)\ \begin{cases} |f(x)-f(y)| \leq \dfrac{C}{V^{1/p}(x,d)} \\ \qquad\qquad \times\max\left\{d^\alpha\|f\|_p^{1-\theta}\|\Delta^{\alpha/2\theta}f\|_p^\theta,\ d^{\alpha'}\|f\|_p^{1-\theta'}\|\Delta^{\alpha'/2\theta'}f\|_p^{\theta'}\right\} \\ \forall\ f\in\mathcal{D},\ x,y\in M. \end{cases}$$

Conversely, (G) for any $p>1$, $\alpha,\alpha'>\frac{\nu}{p}$ and $\theta,\theta'\in]0,1[$, implies (LE), where d denote to $d(x,y)$.

There are several classes of fractal spaces which satisfy the more general form of the heat kernel estimate:

$$\frac{c}{V(x,t^{1/w})}E_w(c,x,y,t) \leq p_t(x,y) \leq \frac{C}{V(x,t^{1/w})}E_w(C,x,y,t),$$

for $t > 0$, $x, y \in M$, where

$$E_w(\lambda, x, y, t) = \exp\left(-\left(\frac{d^w(x,y)}{\lambda t}\right)^{1/(w-1)}\right), \quad \forall\, t, \lambda > 0,\, x, y \in M,$$

and w is a so-called escape time or random walk dimension (see [**1**], [**2**], [**3**], see also [**10**], [**11**], [**12**]). We are also able to write our characterizations of lower heat kernel estimate in this general form.

We choose to write this paper in a more general setting, when M is a metric measure space endowed with a symetric Markov semigroup.

1. Assumptions

In the sequel, we shall place ourselves in a setting similar to the one in [**5**], [**10**], [**13**]: Let (M, d, μ) be a metric measure space endowed with a symmetric Markov semigroup e^{-tA} on $L^2(M, \mu)$ with a measurable kernel p_t, that is

$$e^{-tA} f(x) = \int_M p_t(x,y) f(y)\, d\mu(y), \quad t > 0,\ f \in L^2(M,\mu),\ \mu\text{-a.e. } x \in M.$$

The powers A^α, $\alpha > 0$, of the operator A are densely defined on $L^p(M,\mu)$, $1 \le p < +\infty$, we denote by $\mathcal{D}_p(A^\alpha)$ their domain. Background and more information about this functional setting can be found in [**17**] and references therein. For simplicity, we always write $\mathcal{D}$ for the space $\mathcal{D}_p(A^\alpha)$ required by the context. We assume that $\|e^{-tA} f\|_p \to 0$ as $t \to +\infty$, for all $f \in \mathcal{D}$ and $1 \le p < +\infty$.

We shall assume that for all $x \in M$ and $r > 0$, $0 < V(x,r) < +\infty$ and that the metric space (M, d) satisfies the chain condition: there exists $C > 0$ such that, for all $x, y \in M$, for any $n \in \mathbb{N}^*$, there exists a sequence $\{x_i\}_{i=0}^n$ of points in M such that $x_0 = x$, $x_n = y$ and

$$d(x_i, x_{i+1}) \le C\frac{d(x,y)}{n}, \quad \forall\, i = 0, \dots, n-1.$$

We say that such a sequence is a chain connecting x and y.

Note that these assumptions are well satisfied in the setting of the introduction when M is a connected non-compact complete Riemannian manifold. They will be standing assumptions in this paper and we will refer to them by writing: let (M, d, μ, A) be as above. We recall that a symmetric Markov semigroup on $L^2(M,\mu)$ is analytic on $L^p(M,\mu)$ (see [**21**]).

2. Preliminaries

Notation:

Consider a parameter $w \geq 2$,

$$(UE_w) \qquad p_t(x,y) \leq \frac{C}{V(x,t^{1/w})} E_w(C,x,y,t), \quad \forall\, t>0,\ x,y \in M.$$

$$(LE_w) \qquad \frac{c}{V(x,t^{1/w})} E_w(c,x,y,t) \leq p_t(x,y), \quad \forall\, t>0,\ x,y \in M.$$

$$(DLE_w) \qquad \frac{c}{V(x,t^{1/w})} \leq p_t(x,x), \qquad \forall\, t>0,\ x \in M.$$

$$(LY_w) \quad \frac{c}{V(x,t^{1/w})} E_w(c,x,y,t) \leq p_t(x,y) \leq \frac{C}{V(x,t^{1/w})} E_w(C,x,y,t),$$

for all $t > 0$ and $x, y \in M$.

Note that $(UE_2) = (UE)$ and similarly for the others w-estimations. For all $p > 1$, we denote by p' the conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$. Letters c, C, C' are normally used to denote unimportant positive constants, whose values may change at each occurence. In the sequel, for the sake of simplicity, we sometimes denote $d(x,y)$ by d.

We recall the following proposition. for the proof, in the case $w = 2$, see [**4**, Lemma 1, p. 224], [**7**, §6] or [**20**, Theorem 4.2.8], [**13**, §3.3]; for the general case $w \geq 2$, see [**16**, §4] or [**5**, §3].

Proposition 1. *Assume that (M,d,μ) satisfies the doubling property. If (UE_w) holds, then (DLE_w) holds.*

The following lemma follows from [**10**, Corollary 3.5], [**13**, Lemma 5.1], see also [**5**, p. 801].

Lemma 2. *Assume that (M,d,μ) satisfies the doubling property. If (UE_w) holds and there exist $a, c > 0$ such that*

$$\frac{c}{V(x,t^{1/w})} \leq p_t(x,y), \ \text{for all } t>0,\ x,y \in M \ \text{with } d(x,y) \leq at^{1/w},$$

then (LE_w) holds.

3. Hölder continuity of the heat kernel

At first, let us recall the following proposition, which follows from [**19**, Proposition 3.2] in the case $w = 2$ and [**13**, §5.3] for the general case.

Proposition 3. *Assume that (M,d,μ) satisfies the doubling property. If (LY_w) holds, then there exists $\eta\in]0,1[$ and $C>0$ such that*

$$|p_t(x,z)-p_t(y,z)|\le \frac{C}{V(z,t^{1/w})}\left(\frac{d(x,y)}{t^{1/w}}\right)^{\eta},$$

for all $t>0$ and μ-a.e. $x,y,z\in M$.

In the next result, we give an estimation with a Gaussian factor.

Proposition 4. *Assume that (M,d,μ) satisfies the doubling condition (D_ν). If (LY_w) holds, then there exists $\eta\in]0,1[$ and $C>0$ such that for all $p>1$:*

$$|p_t(x,z)-p_t(y,z)|$$
$$\le C\frac{(E_w(C,x,z,t)+E_w(C,y,z,t))^{1/p}}{V^{1/p}(x,d)V^{1/p'}(z,t^{1/w})}\begin{cases}\left(\dfrac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} & \text{if } d\ge t^{1/w}\\[2ex] \left(\dfrac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}} & \text{if not,}\end{cases}$$

for all $t>0$ and μ-a.e. $x,y,z\in M$, where d denote to $d(x,y)$.

Proof: Let $x,y,z\in M$, $p>1$ and p' its conjugate exponent, one may write

$$|p_t(x,z)-p_t(y,z)|\le|p_t(x,z)-p_t(y,z)|^{1/p'}.(p_t(x,z)+p_t(y,z))^{1/p}.$$

According to Proposition 3 and (UE_w), it follows

$$(1)\quad |p_t(x,z)-p_t(y,z)|\le\frac{C}{V^{1/p'}(z,t^{1/w})}\left(\frac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}}$$
$$\times\left(\frac{1}{V(x,t^{1/w})}E_w(C,x,z,t)+\frac{1}{V(y,t^{1/w})}E_w(C,y,z,t)\right)^{1/p}.$$

Denote to $d(x,y)$ by d. Since $V(x,d)\le V(y,2d)$ and $V(y,2d)\le CV(y,d)$, then $V(x,d)\le CV(y,d)$.

Assume that $d\ge t^{1/w}$. From (D_ν), one has

$$\frac{1}{V(x,t^{1/w})}\le\frac{C}{V(x,d)}\left(\frac{d}{t^{1/w}}\right)^{\nu}$$

and

$$\frac{1}{V(y,t^{1/w})}\le\frac{C}{V(y,d)}\left(\frac{d}{t^{1/w}}\right)^{\nu}\le\frac{C'}{V(x,d)}\left(\frac{d}{t^{1/w}}\right)^{\nu}.$$

Then, (1) yields

$$|p_t(x,z)-p_t(y,z)| \le \frac{C}{V^{1/p}(x,d)}\left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} \frac{1}{V^{1/p'}(z,t^{1/w})} \times \left[E_w(C,x,z,t)+E_w(C,y,z,t)\right]^{1/p}.$$

If $d \le t^{1/w}$. One has also

$$\frac{1}{V(y,t^{1/w})} \le \frac{1}{V(y,d)} \le \frac{C}{V(x,d)}.$$

Analogously, by (1) it follows that

$$|p_t(x,z)-p_t(y,z)| \le \frac{C}{V^{1/p}(x,d)}\left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}} \frac{1}{V^{1/p'}(z,t^{1/w})} \times \left[E_w(C,x,z,t)+E_w(C,y,z,t)\right]^{1/p}. \square$$

Now, we derive an oscillation estimate for the semigroup $(e^{-tA})_{t>0}$.

Proposition 5. *Assume that (M,d,μ) satisfies the doubling condition (D_ν). If (LY_w) holds, then for all $p>1$ there exists $C>0$ such that*

$$|e^{-tA}f(x)-e^{-tA}f(y)| \le \frac{C\|f\|_p}{V^{1/p}(x,d(x,y))}\begin{cases}\left(\dfrac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} & \text{if } d(x,y)\ge t^{1/w}\\ \left(\dfrac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}} & \text{if not,}\end{cases}$$

for all $t>0$, $f\in\mathcal{D}$ and μ-a.e. $x,y\in M$.

Proof: From the definition of the heat kernel, one can write

$$|e^{-tA}f(x)-e^{-tA}f(y)| \le \int_M |(p_t(x,z)-p_t(y,z))f(z)|\,d\mu(z),$$

for all $t>0$, $f\in\mathcal{D}$, $x,y\in M$.

Assume that $d \geq t^{1/w}$. From Proposition 4, it follows

$$\int_M |(p_t(x,z) - p_t(y,z))\, f(z)|\, d\mu(z) \leq \frac{C}{V^{1/p}(x,d)} \left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}}$$
$$\times \int_M \frac{|f(z)|}{V^{1/p'}(z,t^{1/w})} \left(E_w(C,x,z,t) + E_w(C,y,z,t)\right)^{1/p}\, d\mu(z),$$

and by using Hölder inequality, we obtain

$$\text{(2)}\quad \int_M |(p_t(x,z) - p_t(y,z))\, f(z)|\, d\mu(z) \leq \frac{C}{V^{1/p}(x,d)} \left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} \|f\|_p$$
$$\times \left(\int_M \frac{1}{V(z,t^{1/w})} \left(E_w(C,x,z,t) + E_w(C,y,z,t)\right)^{p'/p}\, d\mu(z)\right)^{1/p'}.$$

Let us prove that for all $\gamma > 0$, there is $C > 0$, such that for all $t > 0$.

$$\text{(3)}\quad \int_M \frac{1}{V(z,t^{1/w})} \left(E_w(C,x,z,t) + E_w(C,y,z,t)\right)^{\gamma}\, d\mu(z) \leq C.$$

Indeed, since $(a+b)^\gamma \leq C_\gamma(a^\gamma + b^\gamma)$, for all $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned}(E_w(C,x,z,t) &+ E_w(C,y,z,t))^\gamma \\ &\leq C\left(E_w^\gamma(C,x,z,t) + E_w^\gamma(C,y,z,t)\right) \\ &\leq C\left(E_w(C\gamma^{1-w},x,z,t) + E_w(C\gamma^{1-w},y,z,t)\right).\end{aligned}$$

Let c be the constant in (LE_w). Set $a = \frac{c}{C\gamma^{1-w}}$ and $s = \frac{t}{a}$, then

$$\begin{aligned}E_w(C\gamma^{1-w},x,z,t) &= E_w(c/a,x,z,as) \\ &= E_w(c,x,z,s).\end{aligned}$$

Therefore

$$\int_M \frac{1}{V(z,t^{1/w})} \left(E_w(C,x,z,t) + E_w(C,y,z,t)\right)^{\gamma}\, d\mu(z)$$
$$\leq \int_M \frac{C}{V(z,(as)^{1/w})} \left(E_w(c,x,z,s) + E_w(c,y,z,s)\right)\, d\mu(z).$$

By (D_ν), one has $V(z,\sqrt{as}) \simeq V(z,\sqrt{s})$, $s > 0$. On the other hand, according to (LE_w) and since $(e^{-tA})_{t>0}$ is Markovian, one has

$$\int_M \frac{1}{V(z,\sqrt{s})} E_w(c,x,z,t)\, d\mu(z) \leq \int_M p_s(x,z)\, d\mu(z) \leq 1,$$

for all $s > 0$ and $x \in M$. Therefore, we deduce (3).

By choosing $\gamma = p'/p$, (2) yields

$$|e^{-tA}f(x) - e^{-tA}f(y)| \leq \frac{C}{V^{1/p}(x,d)} \left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} \|f\|_p.$$

If $d \leq t^{1/w}$. Using Proposition 4 and arguing as before, it follows

$$|e^{-tA}f(x) - e^{-tA}f(y)| \leq \frac{C}{V^{1/p}(x,d)} \left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}} \|f\|_p. \qquad \square$$

4. Characterization of the lower gaussian estimate by some Hölder inequalities

4.1. First characterization.

Theorem 6. *Let (M,d,μ,A) as above satisfy the doubling condition (D_ν). Let $w \geq 2$. Assume that (UE_w) holds. Then*

$$(LE_w) \qquad \frac{c}{V(x,t^{1/w})} E_w(c,x,y,t) \leq p_t(x,y), \quad \forall\, t>0, \forall\, x,y \in M,$$

implies that for all p large enough there are $\alpha, \alpha' > \frac{\nu}{p}$ such that

$$(S_w) \quad \begin{cases} |f(x)-f(y)| \leq \dfrac{C}{V^{1/p}(x,d)} \max\left\{d^\alpha \|A^{\alpha/w}f\|_p,\, d^{\alpha'}\|A^{\alpha'/w}f\|_p\right\} \\ \forall\, f \in \mathcal{D},\ x,y \in M. \end{cases}$$

Conversely, (S_w) for any $p>1$ and $\alpha, \alpha' > \frac{\nu}{p}$, implies (LE_w), where d denote to $d(x,y)$.

Proof:

$(\Rightarrow)$ For all $f \in \mathcal{D}$, $t>0$, we know that $\frac{\partial^k}{\partial t^k} e^{-tA} f = A^k e^{-tA} f$ for all $k \in \mathbb{N}$. Then

$$f = \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i e^{-tA} f + \frac{1}{(k-1)!} \int_0^t s^{k-1} A^k e^{-sA} f\, ds, \quad \forall\, k \geq 1.$$

Since $(e^{-tA})_{t>0}$ is analytic, then for all $p>1$

$$\|A^i e^{-tA} f\|_p = \|A^i e^{-(t/2)A} e^{-(t/2)A} f\|_p \leq C t^{-i} \|e^{-(t/2)A} f\|_p.$$

On the other hand, $e^{-tA}f \to 0$, $\forall\, f \in \mathcal{D}$ when $t \to +\infty$, then for all $k \geq 1$:

$$f = \frac{1}{(k-1)!} \int_0^{+\infty} t^{k-1} A^k e^{-tA} f\, dt, \quad \forall\, f \in \mathcal{D}.$$

Hence

$$|f(x)-f(y)| \le C_k \int_0^{+\infty} t^{k-1} |A^k e^{-tA} f(x) - A^k e^{-tA} f(y)|\, ds,$$

for all $k \ge 1$, and $f \in \mathcal{D}$.

By rewriting

$$\begin{aligned}|A^k e^{-tA} f(x) - A^k e^{-tA} f(y)| \\ &= |e^{-(t/2)A} A^k e^{-(t/2)A} f(x) - e^{-(t/2)A} A^k e^{-(t/2)A} f(y)|\end{aligned}$$

and applying Proposition 5 to $A^k e^{-(t/2)A} f$, it follows

$$\begin{aligned}|e^{-tA} f(x) - e^{-tA} f(y)| \\ &\le \frac{C\|A^k e^{-(t/2)A} f\|_p}{V^{1/p}(x, d(x,y))} \begin{cases} \left(\dfrac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} & \text{if } d(x,y) \ge t^{1/w} \\ \left(\dfrac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}} & \text{if not,}\end{cases}\end{aligned}$$

for all $t > 0$ and μ-a.e. $x, y \in M$. Then

$$\begin{aligned}|f(x)-f(y)| \le \frac{C}{V^{1/p}(x,d)} \Bigg(& d^{\frac{\eta}{p'}+\frac{\nu}{p}} \int_0^{d^w} t^{k-\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu}{p})-1} \|A^k e^{-(t/2)A} f\|_p\, dt \\ & + d^{\frac{\eta}{p'}} \int_{d^w}^{+\infty} t^{k-\frac{\eta}{wp'}-1} \|A^k e^{-(t/2)A} f\|_p\, dt \Bigg).\end{aligned}$$

Now, we choose $k > \frac{1}{w}(\frac{\eta}{p'} + \frac{\nu}{p})$ and let $\delta \in]0, \frac{p}{p'}\eta[$.

One can write

$$\|A^k e^{-(t/2)A} f\|_p = \|A^{k-\frac{1}{w}(\frac{\eta}{p}+\frac{\nu+\delta}{p})} e^{-(t/2)A} (A^{\frac{1}{w}(\frac{\eta}{p}+\frac{\nu+\delta}{p})} f)\|_p,$$

then by analyticity, one has

$$\|A^k e^{-(t/2)A} f\|_p \le C_1 t^{\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu+\delta}{p})-k} \|A^{\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu+\delta}{p})} f\|_p.$$

Similarly, one also has

$$\|A^k e^{-(t/2)A} f\|_p \le C_2 t^{\frac{1}{w}(\frac{\eta}{p'}-\frac{\delta}{p})-k} \|A^{\frac{1}{w}(\frac{\eta}{p'}-\frac{\delta}{p})} f\|_p.$$

Then

$$|f(x)-f(y)| \leq \frac{C}{V^{1/p}(x,d)}\Bigg(d^{\frac{\eta}{p'}+\frac{\nu}{p}}\int_0^{d^w} t^{\frac{\delta}{wp}-1}\|A^{\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu+\delta}{p})}f\|_p\,dt$$

$$+\,d^{\frac{\eta}{p'}}\int_{d^w}^{+\infty} t^{-\frac{\delta}{wp}-1}\|A^{\frac{1}{w}(\frac{\eta}{p'}-\frac{\delta}{p})}f\|_p\,dt\Bigg).$$

Therefore

$$|f(x)-f(y)| \leq \frac{C}{V^{1/p}(x,d)}(d^{\alpha}\|A^{\alpha/w}f\|_p + d^{\alpha'}\|A^{\alpha'/w}f\|_p)$$

$$\leq \frac{C}{V^{1/p}(x,d)}\max\{d^{\alpha}\|A^{\alpha/w}f\|_p, d^{\alpha'}\|A^{\alpha'/w}f\|_p\},$$

where $\alpha=\frac{\eta}{p'}+\frac{\nu+\delta}{p}>\frac{\nu}{p}$, $\alpha'=\frac{\eta}{p'}-\frac{\delta}{p}>0$ and $\delta\in]0,\frac{p}{p'}\eta[$.

For $p>\frac{\nu+\eta}{\eta}$, there is $\delta_0>0$ such that $p>\frac{\nu+\eta+\delta}{\eta}$, $\forall\,\delta\in]0,\delta_0[$.

Therefore for $\delta\in]0,\delta_0[$, one has $\eta>\frac{\nu+\eta+\delta}{p}$, then $\eta(1-\frac{1}{p})>\frac{\nu+\delta}{p}$, hence $\alpha'=\frac{\eta}{p'}-\frac{\delta}{p}>\frac{\nu}{p}$.

($\Leftarrow$) Let $p>1$, assume that for some α and $\alpha'>\frac{\nu}{p}$:

$$(S_w)\quad \begin{cases} |f(x)-f(y)|\leq \dfrac{C}{V^{1/p}(x,d)}\max\left\{d^{\alpha}\|A^{\alpha/w}f\|_p,\ d^{\alpha'}\|A^{\alpha'/w}f\|_p\right\} \\ \forall\ f\in\mathcal{D},\ x,y\in M. \end{cases}$$

Let $z\in M$, by analyticity of $(e^{-tA})_{t>0}$, $p_t(.,z)$ belong to $\mathcal{D}_p(A^{\alpha/w})$. Then by choosing $f=p_t(.,z)$ in (S_w), one has for all $x,y\in M$:

$$|p_t(x,z)-p_t(y,z)| \leq \frac{C}{V^{1/p}(x,d)}\max\left\{d^{\alpha}\|A^{\alpha/w}p_t(.,z)\|_p,\ d^{\alpha'}\|A^{\alpha'/w}p_t(.,z)\|_p\right\}.$$

By analyticity, we obtain

$$\|A^{\alpha/w}p_t(.,z)\|_p = \|A^{\alpha/w}e^{-(t/2)A}p_{t/2}(.,z)\|_p \leq Ct^{-\alpha/w}\|p_{t/2}(.,z)\|_p.$$

Since $(e^{-tA})_{t>0}$ is symmetric Markovian, then $\|p_{t/2}(.,z)\|_1\leq 1$, $\forall\,t>0$.

Then from (UE_w) and Hölder inequality, it follows

$$\|p_t(.,z)\|_p \leq \frac{C}{[V(z,t^{1/w})]^{1-\frac{1}{p}}},\quad \forall\,t>0,$$

and by (D_ν), one has

$$\|p_{t/2}(.,z)\|_p \leq \frac{C}{[V(z,t^{1/w})]^{1-\frac{1}{p}}},$$

then

$$\|A^{\alpha/w}p_t(.,z)\|_p \leq C\frac{t^{-\alpha/w}}{[V(z,t^{1/w})]^{1-\frac{1}{p}}}.$$

Therefore

$$|p_t(x,z)-p_t(y,z)|$$

$$\leq \frac{C}{V^{1/p}(x,d)}\max\left\{\left(\frac{d}{t^{1/w}}\right)^{\alpha}\frac{1}{[V(z,t^{1/w})]^{1-\frac{1}{p}}},\left(\frac{d}{t^{1/w}}\right)^{\alpha'}\frac{1}{[V(z,t^{1/w})]^{1-\frac{1}{p}}}\right\}$$

$$\leq \frac{C}{V(z,t^{1/w})}\left(\frac{V(z,t^{1/w})}{V(x,d)}\right)^{1/p}\max\left\{\left(\frac{d}{t^{1/w}}\right)^{\alpha},\left(\frac{d}{t^{1/w}}\right)^{\alpha'}\right\}.$$

By Proposition 1, (UE_w) and (D_ν) yield

$$(DLE_w) \qquad \frac{c}{V(z,t^{1/w})} \leq p_t(z,z), \quad \forall\, z \in M.$$

Thus

$$|p_t(x,z)-p_t(y,z)| \leq C\left(\frac{V(z,t^{1/w})}{V(x,d)}\right)^{1/p}\max\left\{\left(\frac{d}{t^{1/w}}\right)^{\alpha},\left(\frac{d}{t^{1/w}}\right)^{\alpha'}\right\}p_t(z,z).$$

For $z=x$, one has

$$|p_t(x,x)-p_t(y,x)| \leq C\left(\frac{V(x,t^{1/w})}{V(x,d)}\right)^{1/p}\max\left\{\left(\frac{d}{t^{1/w}}\right)^{\alpha},\left(\frac{d}{t^{1/w}}\right)^{\alpha'}\right\}p_t(x,x).$$

If $d \leq t^{1/w}$, (D_ν) yields

$$|p_t(x,x)-p_t(y,x)| \leq C\max\left\{\left(\frac{d}{t^{1/w}}\right)^{\alpha-\frac{\nu}{p}},\left(\frac{d}{t^{1/w}}\right)^{\alpha'-\frac{\nu}{p}}\right\}p_t(x,x).$$

Since α and $\alpha' > \frac{\nu}{p}$, then for a small enough and $d \leq at^{1/w}$

$$|p_t(x,x)-p_t(y,x)| \leq \frac{1}{2}p_t(x,x),$$

consequently

$$p_t(x,y) \geq \frac{1}{2} p_t(x,x) \geq \frac{C}{V(x,t^{1/w})},$$
$$\forall\, x,y \in M, t>0, \text{ such that } d(x,y) \leq at^{1/w}.$$

Therefore, by Lemma 2, one concludes that (LE_w) is satisfied. □

4.2. Second characterization.

Theorem 7. *Let (M,d,μ,A) as above satisfy the doubling condition (D_ν). Let $w \geq 2$. Assume that (UE_w) holds. Then*

$$(LE_w) \qquad \frac{c}{V(x,t^{1/w})} E_w(c,x,y,t) \leq p_t(x,y), \quad \forall\, t>0,\ x,y \in M,$$

implies for all p large enough, there are $\alpha, \alpha' > \frac{\nu}{p}$ and $\theta, \theta' \in]0,1[$ such that $\alpha - \alpha' = \frac{\nu}{p}$, $\frac{\alpha}{\theta} = \frac{\alpha'}{\theta'}$ and

$$(G_w) \begin{cases} |f(x)-f(y)| \leq \dfrac{C}{V^{1/p}(x,d)} \\ \qquad \times \max\left\{ d^\alpha \|f\|_p^{1-\theta} \|A^{\alpha/w\theta} f\|_p^\theta,\ d^{\alpha'} \|f\|_p^{1-\theta'} \|A^{\alpha'/w\theta'} f\|_p^{\theta'} \right\} \\ \forall\, f \in \mathcal{D},\ x,y \in M. \end{cases}$$

Conversely, (G_w) for any $p>1$, $\alpha,\alpha' > \frac{\nu}{p}$ and $\theta,\theta' \in]0,1[$, implies (LE_w), where d denote to $d(x,y)$.

Proof:

($\Rightarrow$) Let $\beta > \frac{\eta}{p'} + \frac{\nu}{p}$. Firstly assume that $\beta < w$. For all $f \in \mathcal{D}$, we know that

$$f = e^{-tA} f + \int_0^t A e^{-sA} f\, ds, \quad \forall\, t>0,$$

then

$$(4) \quad |f(x)-f(y)| \leq |e^{-tA}f(x) - e^{-tA}f(y)| + \int_0^t |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\, ds$$

and

$$\int_0^t |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\, ds = \int_0^t |e^{-(s/2)A}Ae^{-(s/2)A}f(x) - e^{-(s/2)A}Ae^{-(s/2)A}f(y)|\, ds.$$

Assume that $d \geq t^{1/w}$. From Proposition 5, we obtain

$$\int_0^t |e^{-(s/2)A}Ae^{-(s/2)A}f(x) - e^{-(s/2)A}Ae^{-(s/2)A}f(y)|\,ds$$
$$\leq \frac{C}{V^{1/p}(x,d)} d^{\frac{\eta}{p'}+\frac{\nu}{p}} \int_0^t s^{-\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu}{p})} \|Ae^{-(s/2)A}f\|_p\,ds.$$

Since $(e^{-tA})_{t>0}$ is analytic, then $\|Ae^{-(s/2)A}f\|_p \leq Cs^{\frac{\beta}{w}-1}\|A^{\beta/w}f\|_p$. Hence

$$\int_0^t s^{-\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu}{p})}\|Ae^{-(s/2)A}f\|_p\,ds \leq C\|A^{\beta/w}f\|_p \int_0^t s^{-\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu}{p})+\frac{\beta}{w}-1}\,ds$$
$$= Ct^{-\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu}{p})+\frac{\beta}{w}}\|A^{\beta/w}f\|_p.$$

Thus

$$\int_0^t |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\,ds \tag{5}$$
$$\leq \frac{C}{V^{1/p}(x,d)} d^{\frac{\eta}{p'}+\frac{\nu}{p}} t^{-\frac{1}{w}(\frac{\eta}{p'}+\frac{\nu}{p})+\frac{\beta}{w}}\|A^{\beta/w}f\|_p.$$

If $d \leq t^{1/w}$.

$$\int_0^t |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\,ds = \int_0^{d^w} |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\,ds$$
$$+ \int_{d^w}^t |e^{-(s/2)A}Ae^{-(s/2)A}f(x) - e^{-(s/2)A}Ae^{-(s/2)A}f(y)|\,ds.$$

From (5) and Proposition 5, we obtain

$$\int_0^t |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\,ds$$
$$\leq \frac{C}{V^{1/p}(x,d)} d^{\beta}\|A^{\beta/w}f\|_p + \frac{C}{V^{1/p}(x,d)} d^{\frac{\eta}{p'}} \int_{d^w}^t s^{-\frac{\eta}{wp'}}\|Ae^{-(s/2)A}f\|_p\,ds.$$

Since $(e^{-tA})_{t>0}$ is analytic, then

$$\int_{d^w}^t s^{-\frac{\eta}{wp'}}\|Ae^{-(s/2)A}f\|_p\,ds \leq C\|A^{\beta/w}f\|_p \int_{d^w}^t s^{-\frac{\eta}{wp'}+\frac{\beta}{w}-1}\,ds$$
$$= C(t^{-\frac{\eta}{wp'}+\frac{\beta}{w}} - d^{-\frac{\eta}{p'}+\beta})\|A^{\beta/w}f\|_p.$$

Therefore

$$\begin{aligned}
&\int_0^t |Ae^{-sA}f(x) - Ae^{-sA}f(y)|\,ds \\
&\le \frac{C}{V^{1/p}(x,d)} d^\beta \|A^{\beta/w}f\|_p + \frac{C}{V^{1/p}(x,d)} (d^{\frac{\eta}{p'}} t^{-\frac{\eta}{wp'}+\frac{\beta}{w}} - d^\beta)\|A^{\beta/w}f\|_p \\
&\le \frac{C}{V^{1/p}(x,d)} d^{\frac{\eta}{p'}} t^{-\frac{\eta}{wp'}+\frac{\beta}{w}} \|A^{\beta/w}f\|_p. \qquad (6)
\end{aligned}$$

Hence by (4), Proposition 5, (5) and (6), one has for all $t > 0$

$$|f(x) - f(y)| \le \frac{C}{V^{1/p}(x,d)} \times (*),$$

where

$$(*) = \max\left\{ \left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} (\|f\|_p + t^{\beta/w}\|A^{\beta/w}f\|_p), \right.$$
$$\left. \left(\frac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}} (\|f\|_p + t^{\beta/w}\|A^{\beta/w}f\|_p) \right\}.$$

By choosing $t = (\frac{\|f\|_p}{\|A^{\beta/w}f\|_p})^{w/\beta}$, one obtains

$$\begin{aligned}
|f(x) - f(y)| \le{}& \frac{C}{V^{1/p}(x,d)} \\
&\times \max\left\{ d^{\frac{\eta}{p'}+\frac{\nu}{p}} \|f\|_p^{1-\theta}\|A^{\beta/w}f\|_p^\theta,\, d^{\frac{\eta}{p'}}\|f\|_p^{1-\theta'}\|A^{\beta/w}f\|_p^{\theta'} \right\},
\end{aligned}$$

where $\theta = \frac{1}{\beta}(\frac{\eta}{p'} + \frac{\nu}{p})$ and $\theta' = \frac{\eta}{\beta p'}$.

Now, if $\beta \ge w$. Fix $k > \frac{\beta}{w}$, we know that

$$f = \sum_{i=0}^{k-1} \frac{t^i}{i!} A^i e^{-tA} f + \frac{1}{(k-1)!} \int_0^t s^{k-1} A^k e^{-sA} f\,ds, \quad \forall\, t > 0.$$

By Proposition 5 and the analyticity, it yields, for all $i = 0, \dots, k-1$:

$$\begin{aligned}
|A^i e^{-tA}f(x) - A^i e^{-tA}f(y)| &\le \frac{C}{V^{1/p}(x, d(x,y))} \|A^i e^{-(t/2)A} f\|_p \times (**) \\
&\le \frac{Ct^{-i}}{V^{1/p}(x, d(x,y))} \|f\|_p \times (**)
\end{aligned}$$

for all $t > 0$, $f \in \mathcal{D}$ and μ-a.e. $x, y \in M$. where

$$(**) = \begin{cases} \left(\dfrac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}} & \text{if } d(x,y) \geq t^{1/w} \\ \left(\dfrac{d(x,y)}{t^{1/w}}\right)^{\frac{\eta}{p'}} & \text{if not.} \end{cases}$$

On the other hand, one can write

$$\begin{aligned} &\int_0^t s^{k-1}(A^k e^{-sA} f(x) - A^k e^{-sA} f(y))\, ds \\ &= \int_0^t s^{k-1}(e^{-(s/2)A} A^k e^{-(s/2)A} f(x) - e^{-(s/2)A} A^k e^{-(s/2)A} f(y))\, ds, \end{aligned}$$

and by analyticity, we have

$$\|A^k e^{-(s/2)A} f\|_p \leq C s^{\frac{\beta}{w}-k} \|A^{\beta/w} f\|_p.$$

Then, by using Proposition 5 and arguing similarly as the first case, we also obtain

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{C}{V^{1/p}(x,d)} \\ &\times \max\left\{ d^{\frac{\eta}{p'}+\frac{\nu}{p}} \|f\|_p^{1-\theta} \|A^{\beta/w} f\|_p^{\theta},\, d^{\frac{\eta}{p'}} \|f\|_p^{1-\theta'} \|A^{\beta/w} f\|_p^{\theta'} \right\}, \end{aligned}$$

where $\theta = \frac{1}{\beta}(\frac{\eta}{p'} + \frac{\nu}{p})$ and $\theta' = \frac{\eta}{\beta p'}$.

Set $\alpha = \frac{\eta}{p'} + \frac{\nu}{p}$ and $\alpha' = \frac{\eta}{p'}$, then $\beta = \frac{\theta}{\alpha} = \frac{\theta'}{\alpha'}$, $\alpha - \alpha' = \frac{\nu}{p}$, $\alpha > \frac{\nu}{p}$, $\alpha' > 0$ and

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{C}{V^{1/p}(x,d)} \\ &\times \max\left\{ d^{\alpha} \|f\|_p^{1-\theta} \|A^{\alpha/w\theta} f\|_p^{\theta},\, d^{\alpha'} \|f\|_p^{1-\theta'} \|A^{\alpha'/w\theta'} f\|_p^{\theta'} \right\}. \end{aligned}$$

For $p > \frac{\nu+\eta}{\eta}$, one has $\eta > \frac{\nu+\eta}{p}$, then $\eta(1 - \frac{1}{p}) > \frac{\nu}{p}$, hence $\alpha' = \frac{\eta}{p'} > \frac{\nu}{p}$.

($\Leftarrow$) Let $p > 1$, assume that for some $\alpha, \alpha' > \frac{\nu}{p}$ and $\theta, \theta' \in]0, 1[$:

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{C}{V^{1/p}(x,d)} \\ &\times \max\left\{ d^{\alpha} \|f\|_p^{1-\theta} \|A^{\alpha/w\theta} f\|_p^{\theta},\, d^{\alpha'} \|f\|_p^{1-\theta'} \|A^{\alpha'/w\theta'} f\|_p^{\theta'} \right\}, \end{aligned}$$

for all $f \in \mathcal{D}$ and $x, y \in M$.

Let $z \in M$, by analyticity of $(e^{-tA})_{t>0}$, $p_t(.,z)$ belong to $\mathcal{D}_p(A^{\alpha/w})$. Then by choosing $f = p_t(.,z)$ in the previous inequality, one has for all $x, y \in M$:

$$|p_t(x,z) - p_t(y,z)| \leq \frac{C}{V^{1/p}(x,d)} \times \max\Big\{ d^\alpha \|p_t(.,z)\|_p^{1-\theta} \|A^{\alpha/w\theta} p_t(.,z)\|_p^\theta, \\ d^{\alpha'} \|p_t(.,z)\|_p^{1-\theta'} \|A^{\alpha'/w\theta'} p_t(.,z)\|_p^{\theta'} \Big\}.$$

By analyticity,

$$\|A^{\alpha/w\theta} p_t(.,z)\|_p = \|A^{\alpha/w\theta} e^{-(t/2)A} p_{t/2}(.,z)\|_p \leq C t^{-\alpha/w\theta} \|p_{t/2}(.,z)\|_p.$$

Since $(e^{-tA})_{t>0}$ is symmetric Markovian, then $\|p_{t/2}(.,z)\|_1 \leq 1$. Hence (UE_w) yields

$$\|p_t(.,z)\|_p \leq \frac{C}{[V(z,t^{1/w})]^{1-\frac{1}{p}}}, \quad \forall\, t > 0,$$

and by (D_ν), one has

$$\|p_{t/2}(.,z)\|_p \leq \frac{C}{[V(z,t^{1/w})]^{1-\frac{1}{p}}},$$

then

$$\|p_t(.,z)\|_p^{1-\theta} \|A^{\alpha/w\theta} p_t(.,z)\|_p^\theta \leq C \frac{t^{\alpha/w}}{[V(z,t^{1/w})]^{1-\frac{1}{p}}}.$$

Therefore

$$|p_t(x,z) - p_t(y,z)| \\ \leq \frac{C}{V^{1/p}(x,d)} \max\left\{ \left(\frac{d}{t^{1/w}}\right)^\alpha \frac{1}{[V(z,t^{1/w})]^{1-\frac{1}{p}}}, \left(\frac{d}{t^{1/w}}\right)^{\alpha'} \frac{1}{[V(z,t^{1/w})]^{1-\frac{1}{p}}} \right\} \\ \leq \frac{C}{V(z,t^{1/w})} \left(\frac{V(z,t^{1/w})}{V(x,d)}\right)^{1/p} \max\left\{ \left(\frac{d}{t^{1/w}}\right)^\alpha, \left(\frac{d}{t^{1/w}}\right)^{\alpha'} \right\}.$$

Then, similarly as in the proof of the converse in Theorem 6, we obtain the desired result. □

A consequence of Theorems 6 and 7 is the following.

Corollary 8. *Let (M, d, μ, A) as above satisfy the doubling condition (D_ν). Assume that (UE_w) holds. Then*

$$(S_w) \text{ for any } p > 1 \text{ and } \alpha, \alpha' > \frac{\nu}{p},$$

implies, for all p large enough, that there are $\alpha, \alpha' > \frac{\nu}{p}$ and $\theta, \theta' \in]0, 1[$ such that $\alpha - \alpha' = \frac{\nu}{p}$, $\frac{\alpha}{\theta} = \frac{\alpha'}{\theta'}$ and (G_w) holds.

Conversely, (G_w) for any $p > 1$ and $\alpha, \alpha' > \frac{\nu}{p}$, $\theta, \theta' \in]0, 1[$, implies, for all p large enough, that there are $\alpha, \alpha' > \frac{\nu}{p}$ such that (S_w) holds.

Remark 9. We can also deduce (G_w) from (S_w) by using the momentum inequality: for a nonnegative operator in a Banach space

$$\|A^{\alpha/w} f\|_p \leq C\|f\|_p^{1-\theta} \|A^{\alpha/w\theta} f\|_p^\theta,$$

(see [**15**]). The previous corollary, gives us more information about the relation between α, α', θ and θ'.

4.3. Further results.

Let (M, d, μ, A) be as before and $w \geq 2$. From the doubling property, one can also write, there are $\nu > 0$ and $\nu' \geq 0$ such that

$$(D_{\nu,\nu'}) \qquad C' \left(\frac{s}{r}\right)^{\nu'} \leq \frac{V(x, s)}{V(x, r)} \leq C\left(\frac{s}{r}\right)^\nu, \quad \forall\, x \in M,\ s \geq r > 0.$$

Note that, when M has an infinite measure, we can take $\nu' > 0$, (see [**8**]).

By using $(D_{\nu,\nu'})$ instead of (D_ν) and arguing as before, almost all the previous results can be written for all $p > 1$ with ν and ν'. In other words, one has just take $\nu' = 0$ (which is always possible) to find the previous proofs.

More precisely, instead of Propositions 4 and 5, we obtain: If (LY_w) holds, then there exists $\eta \in]0, 1[$ such that for all $p > 1$:

$$|p_t(x, z) - p_t(y, z)|$$
$$\leq C \frac{(E_w(C, x, z, t) + E_w(C, y, z, t))^{1/p}}{V^{1/p}(x, d) V^{1/p'}(z, t^{1/w})} \begin{cases} \left(\dfrac{d}{t^{1/w}}\right)^{\frac{\eta}{p'} + \frac{\nu}{p}} & \text{if } d \geq t^{1/w} \\ \left(\dfrac{d}{t^{1/w}}\right)^{\frac{\eta}{p'} + \frac{\nu'}{p}} & \text{if not,} \end{cases}$$

for all $t > 0$ and μ-a.e. $x, y, z \in M$.

And

$$(7)\quad |e^{-tA}f(x)-e^{-tA}f(y)| \leq \frac{C\|f\|_p}{V^{1/p}(x,d)} \begin{cases} \left(\dfrac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu}{p}}, & \text{if } d \geq t^{1/w} \\ \left(\dfrac{d}{t^{1/w}}\right)^{\frac{\eta}{p'}+\frac{\nu'}{p}} & \text{if not,} \end{cases}$$

for all $t>0$, $f \in \mathcal{D}$ and μ-a.e. $x, y \in M$, where d denote to $d(x,y)$.

Consequently, we obtain the following proposition.

Proposition 10. *Let (M,d,μ,A) as before satisfy the doubling condition $(D_{\nu,\nu'})$, assume that the (UE_w) holds, then for all $p>1$,*

(i) $\quad (LE_w) \Rightarrow (S_w)$ *for some* $\alpha > \dfrac{\nu}{p}$ *and* $\alpha' > \dfrac{\nu'}{p}$.

(ii) $\quad (LE_w) \Rightarrow (G_w)$ *for some* $\alpha > \dfrac{\nu}{p}$, $\alpha' > \dfrac{\nu'}{p}$

such that $\alpha - \alpha' = \dfrac{\nu-\nu'}{p}$ *and* $\dfrac{\alpha}{\theta} = \dfrac{\alpha'}{\theta'}$.

Proof: (i) By using (7) instead of Proposition 5 in the proof of Theorem 6, we obtain for all $p>1$, (S_w) holds for $\alpha = \frac{\eta}{p'} + \frac{\nu+\delta}{p}$ and $\alpha' = \frac{\eta}{p'} + \frac{\nu-\delta}{p}$ where $\delta \in]0, \frac{p}{p'}\eta[$, whence $\alpha > \frac{\nu}{p}$ and $\alpha' > \frac{\nu'}{p}$.

(ii) Similarly as in the proof of Theorem 7 and by using (7) instead of Proposition 5, we obtain that for all $p>1$, there are $\alpha = \frac{\eta}{p'} + \frac{\nu}{p}$, $\alpha' = \frac{\eta}{p'} + \frac{\nu'}{p}$, $\theta, \theta' \in]0,1[$ such that $\frac{\alpha}{\theta} = \frac{\alpha'}{\theta'}$ and (G_w) holds. It is clear that $\alpha - \alpha' = \frac{\nu-\nu'}{p}$. ☐

In the case when the volume growth is polynomial (i.e. $V(x,r) \simeq r^D$, $D>0$), we reobtain the following result from Ouhabaz (see [**16**]).

Corollary 11. *Assume that $V(x,r) \simeq r^D$, if (UE_w) holds, then for all $p>1$*

$$(LE_w) \quad \Leftrightarrow \quad \frac{|f(x)-f(y)|}{d(x,y)^{\alpha-\frac{D}{p}}} \leq C\|f\|_p^\theta \|A^{\alpha/w\theta} f\|_p^{1-\theta},$$

$$\textit{for some } \alpha > \frac{D}{p},\ \theta \in]0,1[.$$

Proof: If $V(x,r) \simeq r^D$, then one can write $(D_{\nu,\nu'})$ with $\nu = \nu' = D$. Therefore

($\Leftarrow$) it suffices to take $\alpha = \alpha'$ and use Theorem 7.

($\Rightarrow$) by applying (ii) of the previous proposition, (LE_w) implies (G_w) with $\alpha = \alpha' > \frac{D}{p}$ and $\theta = \theta'$. □

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